Plate Theory for Metric-Constrained Actuation of Liquid Crystal Elastomer Sheets

Lucas Bouck
Carnegie Mellon University

Liquid crystal elastomers (LCEs) combine the large deformation response of a cross-linked polymer network with the nematic order of liquid crystals pendent to the network. The focus of this talk is the actuation of LCE sheets where the nematic order, modeled by a director field, is specified heterogeneously in the plane of the sheet. Heating such a sheet leads to a large spontaneous deformation, coupled to the director design through a well-known metric constraint. We derive a plate theory for the pure bending deformations of patterned LCE sheets in the limit that the sheet thickness tends to zero using the framework of Γ-convergence. After dividing the bulk energy by the cube of the thickness to set a bending scale, we show that all limiting midplane deformations with bounded energy at this scale satisfy the aforementioned metric constraint. We then identify the energy of our plate theory as an ansatz-free lower bound of the limit of the scaled bulk energy, and construct a recovery sequence that achieves this plate energy for all smooth enough midplane deformations. We then apply the plate theory to some known examples. This is joint work with David Padilla-Garza (Einstein Institute of Mathematics) and Paul Plucinsky (University of Southern California).

Internal constraints and dimension reduction of deformations

Yi-chao Chen
University of Houston

A material is subject to an internal constraint if the class of possible deformations is limited a priori at interior points of the material. Some well-known internal constraints for bulk materials include incompressibility, inextensibility, and rigidity.

A common scheme in deriving the equations of equilibrium of a constrained material by variational methods is to introduce the constraint function in the energy functional as a Lagrange multiplier, which leads to a part of the stress that is indeterminate by the constitutive function of the material.

An alternative scheme is to "solve" the constraint to obtain a representation of the class of possible deformations. This class often has a reduced dimension comparing to the class of deformations of an unconstrained material.

In this talk, we examine the material surfaces subject to the constraint of inextensibility. It is found that the constraint of inextensibility reduces the dimension of the material surface from two to one. In other words, there is a one-to-one correspondence between the deformations of an inextensible material surface and the deformations of a material line. This effectively reduces the energy functional from a surface integral to a line integral. The resulting Euler-Lagrange equation consists of a system of ordinary equations, instead of a system of partial differential equations of two variables when the constraint is included in the energy functional as a Lagrange multiplier.

A Quasi-Continuum Framework for Rate-Dependent Fracture in Polymer-like Networks

Ahmed Elbanna and Ahmed Ghareeb

Department of Civil and Environmental Engineering,
University of Illinois Urbana Champaign

The mechanical behavior of soft polymeric materials is inherently rate-dependent, with their stiffness, strength, and fracture characteristics potentially influenced by loading rates. This study extends the quasi-continuum (QC) approach previously developed for elastic polymer networks to incorporate rate-dependent effects, including viscoelasticity at the polymer chain level and dynamic bond breakage. The QC method is a multiscale computational framework that reduces the computational cost of fully discrete simulations by adaptively coarse-graining the polymer network, while retaining fine-scale resolution in regions of high deformation, such as crack tips. A novel homogenization rule is introduced for the viscous forces in polymer chains, enabling an accurate transition from microscale to macroscale representation while preserving localized fracture phenomena. The adaptive QC framework is implemented within a nonlinear finite element setting, allowing for efficient resolution of crack propagation without predefining fracture energy or length scales.

Verification against fully discrete simulations demonstrates the accuracy of the method in capturing viscoelastic loading/unloading behavior, creep-recovery response, and rate-dependent failure of polymer networks. The study explores the effects of stretch rate, chain viscoelasticity, and dynamic bond breakage on fracture characteristics, revealing four distinct fracture modes: crack blunting with bulk damage, steady-state crack propagation, crack arrest, and non-steady-state crack propagation characterized by sharp cracks. Statistical analysis confirms that the peak force and energy dissipation scale differently for networks dominated by viscoelasticity versus those governed by dynamic bond breakage, with the former exhibiting weak scaling with stretch rate and the latter following a logarithmic trend. Specifically, dynamic bonds become stronger at higher rates allowing the chains to stretch further and explore the strongly nonlinear portion of the force stretch curve thus leading to higher peak strength.

The proposed method provides a computationally efficient yet a robust and physically consistent representation of fracture in rate-dependent polymer networks, bridging the gap between continuum and fully discrete models. The results contribute to fundamental understanding of rate-dependent fracture processes and open avenues for designing polymeric materials with enhanced mechanical performance.

Symmetries of the Elasticity Tensor Induced by the Jaumann Rate of the Kirchhoff Stress

Salvatore Federico[1] and Patrizio Neff[2]

[1]*Dept. Mechanical and Manufacturing Engineering, The University of Calgary, Canada*

[2]*Faculty of Mathematics, University of Duisburg-Essen, Germany*

salvatore.federico@ucalgary.ca, patrizio.neff@uni-due.de

In Continuum Mechanics and in Non-Linear Elasticity, the need for *objective* rates arises from the fact that the regular substantial time derivative is not frame-indifferent, i.e., it is not invariant under time-dependent changes of the spatial reference frame. Objectivity means more than frame-indifference: objective rates are indeed invariant under diffeomorphisms [3], i.e., invertible differentiable maps that admit a differential inverse.

The Jaumann rate (also known as Zaremba-Jaumann rate [8, 7]) is extensively used to define stress rates in Computational Mechanics because it is not only objective, but also *corotational* with respect to the spin tensor, the skew-symmetric part of the velocity gradient. A time-dependent basis is called corotational with respect to a skew-symmetric tensor if the time derivative of each basis vector is obtained locally via Poisson's theorem, i.e., by applying the skew-symmetric tensor on the basis vector as a linear map. In a corotational basis, the components of the associated corotational objective rate equal the components of the regular substantial time derivative. This is the case for the Jaumann rate, associated with the spin tensor and the Green-Naghdi rate, associated with the rigid spin tensor [4, 5, 6].

This work (Federico et al., 2025) focusses on the Jaumann rate and has three main objectives. First, we review the properties of the Jaumann rate in a completely covariant setting, and show its relation with the Lie derivative [3, 1]. Second, we study the symmetries of the elasticity tensor induced by the Jaumann rate of the Kirchhoff stress. This tensor possesses both minor symmetries (i.e., invariance under exchange of the indices in the first pair and the indices in the second pair, separately) and the major symmetry (invariance under exchange of the first pair of indices with the second), which was missed in some past work [6]. In contrast, the elasticity tensor induced by the Jaumann rate of the Cauchy stress does not possess major symmetry. This is the reason why the Jaumann rate of the Kirchhoff stress is preferred. Third, we provide an absolute expression of this elasticity tensors, in terms of an expression not involving contractions with arbitrary second-order tensors. This absolute expression, both in component-free form and component form, is particularly useful for numerical applications.

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Electro- and Magneto-Elastic Modeling with Applications to Potassium Sodium Niobate and Iron-Nickel Alloys

Georgios Grekas

Computer, Electrical, Mathematical Sciences and Engineering Division,
King Abdul- lah University of Science and Technology (KAUST),
Thuwal, Saudi Arabia

Ferroelectric and ferromagnetic materials exhibit complex multiscale behaviors driven by nonlinear electromechanical and magnetoelastic interactions, posing significant challenges in understanding their phase transitions and domain structures. In the ferroelectric material potassium sodium niobate (KNN)—a promising lead-free alternative-cubic-tetragonal-orthorhombic transformations and intermediate twinning have been observed. We present a geometrically nonlinear electroelastic model that successfully predicts experimentally observed microstructures, including laminates, crossing twins, intermediate twinning, and spontaneous polarization as energy minimizing states. Similarly, in ferromagnetic materials, we study magnetic domain evolution developing a numerical implementation of micromagnetics theory. This theory represents a partially linearized version of the full nonlinear magnetoelastic energy. With minimal assumptions, we demonstrate, for a soft magnetic material like FeNi, that magnetostriction affects coercivity and the evolution of magnetic domain patterns under an applied field at the nanoscale, improving the agreement between experiments and theory.

Internal constraints in uniaxial nematic elastomers: Indeterminacy and resolution via gauge relations

Pastore A.[1], Grillo A.[1], Fried E.[2]

[1] *Dipartimento di Scienze Matematiche (DISMA) "G. L. Lagrange", Politecnico di Torino*
[2] *Mechanics and Materials Unit, Okinawa Institute of Science and Technology (OIST)*
{alfio.grillo, andrea.pastore}@polito.it; *eliot.fried@oist.jp*

We present a variational formulation for uniaxial nematic elastomers in equilibrium, subject to three internal constraints: (i) isochoricity of the deformation, reflecting the incompressibility of the underlying polymer network; (ii) inextensibility of the director, corresponding to its characterization as a unit vector field; and (iii) vanishing gradient of the inextensibility condition. We enforce constraints (i)–(iii) via the method of Lagrange multipliers and determine the equilibrium equations of the system by applying the principle of virtual work, subject to the requirement that the virtual work performed by the generalized reaction forces on the virtual displacements of the system vanishes. This procedure results in a formulation wherein the Lagrange multipliers associated with constraints (ii) and (iii) cannot be uniquely determined from the equilibrium equations alone. To resolve this indeterminacy, we propose two equivalent strategies. The first hinges on requiring that the Lagrange multipliers associated with constraints (ii) and (iii) satisfy appropriate gauge conditions. The second involves introducing two effective Lagrange multipliers that can be uniquely determined and replace the original ones in the formulation of the problem. We also examine some physical and computational implications of these alternative formulations.

The inverse-deformation approach to fracture

Arnav Gupta
Field of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, USA
E-mail: ag2592@cornell.edu

Understanding spontaneous brittle fracture in elastic materials has great theoretical and practical importance in many fields, including engineering design and safety, theoretical mechanics, biomechanics and material science. The possibility of discontinuous displacement due to fracture makes the problem difficult to analyze rigorously and to compute reliably. In this talk, we discuss the inverse-deformation approach to fracture our group has developed. This method overcomes the problem of discontinuous deformations without using damage/phase-field variables or pre-existing cracks. Instead of the deformation, we solve for the inverse deformation mapping (from the deformed to the reference configuration). This can be extended to be continuous and piecewise smooth, even when the original deformation has discontinuities describing cracks. We propose a non-convex constitutive law with the qualitative form of a Lennard-Jones potential. Under the inverse transformation due to Shield and Carlson, the energy naturally transforms to a two-well potential; the two wells correspond to the unbroken and broken (vacuum) states. We add a higher-gradient term to the inverse energy, which gives rise to surface energy of new crack surfaces. The opened cracks occur at locations where the vacuum phase appears. We provide existence theorems for energy minimizers and a rigorous weak formulation that allows for fractured solutions. The analysis of the problem is difficult due to the presence of a nonlinear inequality constraint allowing fracture but prohibiting orientation reversal. We obtain numerical results for fracture in a 2D rectangular domain using FEM-based numerical bifurcation/continuation and the active-set method for exact imposition of the inequality constraint. Spontaneous nucleation of cracks occurs as a global bifurcation from the homogeneously deformed state at a critical load. On further stretching, the deformed cracks open and grow through motion of the crack tips. The crack faces precisely delineate the material (unbroken) phase from the broken phase (vacuum). Furthermore, the inverse-deformation maps the broken region in the deformed configuration to a curve in the reference, revealing that the actual deformation is sharply discontinuous. Computed cracks are totally empty of matter, as opposed to approximate damage zones with low density resulting from phase field models.

Shear stresses in fluid and solid membranes with bending elasticity

James Hanna
University of Nevada

with

Sanjay Dharmavaram
Bucknell University

The topic will be addressed with a focus on two simple models arising in soft matter. We illustrate how shear stresses can arise from a bending energy through a coupling between curvature and surface stresses, a feature incidental to the fluid or solid nature of the membrane. The resistance to tangential flows on the surface is distinct from the ability to support tangential stresses. A fluid-like Helfrich bending energy contributes shear stresses, while a related solid-like energy, the correct continuum limit of a Seung Nelson discrete bending term, produces a stress tensor with purely isotropic tangential part. We will relate the latter fact to recent work on the bending response of plates. We will also discuss the balance of pseudomomentum and comment on a few things we found in the literature, including consequences for pulled tether formation in lipid bilayers.

ABSTRACT
ON THE KINEMATICS AND STATICS OF MULTIPOLAR MEDIA

VLADIMIR GOL'DSHTEIN AND REUVEN SEGEV

We consider the distribution of a multipolar medium in the physical space, the force distribution acting on the medium, and the transport of the medium. This applies but is not limited to electrostatics. The physical theory is based on the assumption that there is some extensive property Π, e.g., the electric charge, for which there is a potential field ϕ. Modeling the physical space, $\mathcal{S}$, by $\mathbb{R}^3$, a potential field is assumed to be a smooth real-valued function of compact support—a test function—and the class of all potential fields, or variations thereof, is denoted as $\mathcal{D}$.

A distribution of an r-multipolar medium in space is a Schwartz distribution, Q, in the dual space $\mathcal{D}^*$, specified by a given collection of Borel measures $q^{i_1\dots i_k}$, $0 \leqslant k \leqslant r$, in the form

$$Q(\phi) = \sum_{0\leqslant k\leqslant r} \int_{\mathbb{R}^3} \phi_{,i_1\dots i_k}\, dq^{i_1\dots i_k}, \tag{1}$$

where the summation convention is used and a comma subscript indicates partial differentiation.

When ϕ is interpreted as a variation of the potential field, $Q(\phi)$ is interpreted as the resulting variation of potential energy. The representation by measures allows the description of a concentrated charge or a single dipole, as well as smooth distributions of multipoles, such as in a dielectric medium. For smooth distributions, there are smooth functions $\rho^{i_1\dots i_k}$, $0 \leqslant k \leqslant r$, such that

$$Q(\phi) = \sum_{0\leqslant k\leqslant r} \int_{\mathbb{R}^3} \phi_{,i_1\dots i_k} \rho^{i_1\dots i_k}\, dV\,. \tag{2}$$

In addition, the representation by measures enables the restriction of the functional Q to Borel measurable subsets $\mathcal{B}$ to obtain functionals

$$Q_{\mathcal{B}}(\phi) := \sum_{0\leqslant k\leqslant r} \int_{\mathcal{B}} \phi_{,i_1\dots i_k}\, dq^{i_1\dots i_k}\,. \tag{3}$$

For example, assuming that the subset $\mathcal{B}$ is sufficiently regular so that the Green-Gauss theorem is applicable, the bound charge and the bound polarization induced by a smooth quadrupole distribution are computed. In particular, a charge density per unit length is induced along the edges in $\partial\mathcal{B}$.

A motion of a multipole distribution is modeled by a one-parameter group of diffeomorphisms of $\mathbb{R}^3$. For a fixed potential distribution, a variation of the potential energy induced by velocity field, v, of the property, is equal to minus the power of the field in the form

$$P_{\mathcal{B}} = F_{\mathcal{B}}(v) = -\sum_{0\leqslant k\leqslant r} \int_{\mathcal{B}} \phi_{,i_1\dots i_k i_{k+1}} v^{i_{k+1}}\, dq^{i_1\dots i_k}\,. \tag{4}$$

It is noted that by Equation (4), the velocity field enables a k-th order component of a multipole to sample the $(k+1)$-partial derivatives of the potential as if it were a $(k+1)$-th component of an $(r+1)$-multipolar distribution. For example, a moving charge distribution samples the derivatives of the potential function as

if it were a dipole distribution. Since the power is linear in v, it is the result of applying a force functional, $F_{\mathcal{B}}$, to the velocity field. Thus, one can compute the forces and (hyper-) stresses that the field exerts on the multipolar medium.

A related subject is that of (hyper-) fluxes of order r. Classically, the balance of an extensive property is expressed by the balance differential equation for its flux field u as $u^i_{,i} + \beta = s$ and boundary condition $\tau = u \cdot \nu$, where β is the time rate of the density, s is the source, and τ is the flux density on the boundary, and ν is the unit normal to the boundary. These may be replaced by the variational form

$$P_{\mathcal{B}} = \int_{\partial\mathcal{B}} \tau\phi \, dA + \int_{\mathcal{B}} \beta\phi \, dV = \int_{\mathcal{B}} s\phi \, dV + \int_{\mathcal{B}} u^i \phi_{,i} \, dV, \tag{5}$$

for any test function, $\phi \in \mathcal{D}$, which may be interpreted as a potential function. Hence, $P_{\mathcal{B}}$ may be interpreted as the power associated with the transfer of the property. The last equation may be generalized to

$$P_{\mathcal{B}} = \Phi_{\mathcal{B}}(\phi) = \sum_{0 \leqslant k \leqslant r} \int_{\mathcal{B}} \phi_{,i_1 \dots i_k} \, d\mathfrak{s}^{i_1 \dots i_k}, \tag{6}$$

for a given collection of Borel measures $\mathfrak{s}^{i_1 \dots i_k}$, $0 \leqslant k \leqslant r$. The functional Φ is referred to as a hyperflux of order r. In the smooth case, the measures $\mathfrak{s}^{i_1 \dots i_k}$ are represented by smooth densities, $s^{i_1 \dots i_k}$, relative to the volume measure. For example, we show that a moving polarized medium induces fluxes per unit length of the edges of a region.

Finally, we discuss the generalization of all the above to the case where the physical space is taken as a general differentiable manifold. While the basic definitions may be extended to manifolds using the notion of the jet of a function, in general, one cannot isolate the components of a given order k, invariantly, that is the collections, $\phi_{,i_1 \dots i_k}$ or $\mathfrak{s}^{i_1 \dots i_k}$.

Ben-Gurion University of the Negev
Email address: rsegev@post.bgu.ac.il

Isometric immersions and framed curves

Brian Seguin

Loyola University Chicago

bseguin@luc.edu

The work presented in this talk is motivated by the following problem in mechanics: what is the equilibrium shape of an unstretchable, elastic surface subject to prescribed edge conditions? If the elastic surface starts as a rectangular strip and is deformed to obtain a Möbius band, Wunderlich realized that the shape of the deformed surface and its bending energy are determined by a framed curve. Invoking an energy minimization principle, he then reasoned without proof that the equilibrium shape of the deformed surface can be obtained by minimizing the energy over all such framed curves. In this talk, it will be shown that Wunderlich's idea applies to a simply connected, planar reference region with corners that undergoes a C^2 isometric immersion. Additionally, it will be shown that every framed curve that satisfied prescribed conditions, each with a clear geometric interpretation, determines an isometric immersion that is C^2 almost everywhere. This is joint work with Eliot Fried.